Part I

1. (a) (10 pts) Please derive the homogeneous vector Helmholtz equation in a simple and nonconducting source-free medium from the following Maxwell's equations.

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{H} = 0$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times H = j\omega\varepsilon\vec{E}$$

(b) (5 pts) Calculate the wave number ***k*** associated with the wave described by the Helmholtz equation in (a).

(c) (5 pts) Consider a uniform plane wave characterized by a uniform E_x, write the one-dimensional wave equation for E_x.

(d) (5 pts) A spherical wave can be described by $U(r) = \frac{A}{r}e^{ikr}$. Verify that this is a solution of the homogeneous Helmholtz equation everywhere except at the origin.

Note that Laplacian in spherical coordinates can be written as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{(r\sin\theta)^2}\frac{\partial^2}{\partial\phi^2}$$

(e) (5 pts) Why cannot you extend your method to show that $U(r)$ is a solution to the Helmholtz equation at the origin? Do you think it is or isn't a solution there?

2. (10 pts) Show that the instantaneous Poynting vector of a circularly polarized plane wave propagating in a lossless medium is a constant that is independent of time and distance.
3. (10 pts) An oil film that has a refractive index n_{oil}=1.40 floats on a smooth water surface, which has n_w=1.33. It reflects most strongly at 672 nm red wavelength and appears to have no reflection at the 504 nm blue wavelength. What is the minimum thickness of the oil film?

Part II

4. (4 pts) Explain why single-conductor hollow waveguides cannot support TEM waves.
5. (6 pts) Starting from two time-harmonic Maxwell's curl equation, express the transverse electric field component E_x and E_y in terms of the longitudinal components E_z and H_z.
6. In a dielectric-slab waveguide of thickness d, let ε_d and μ_d be the permittivity and permeability of dielectric slab, respectively, as show in Figure 1.
   (a) (14 pts) Assuming this problem with no dependence on the x coordinate, find the field solution of TM waves.
   (b) (6 pts) Find the dispersion relations and cutoff frequencies of TM waves.

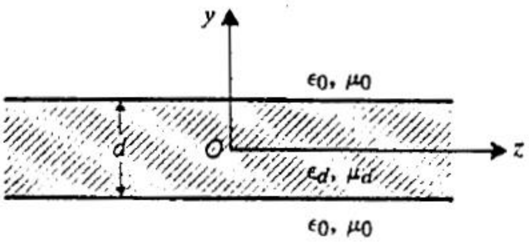

Figure 1

7. Considering a rectangular waveguide with both ends closed by a conducting wall and z axis chosen as the reference "direction of propagation", the interior dimensions of the cavity are a, b, and d in x, y, and z coordinates, respectively. We can obtained the field components of TE_{mnp} modes:

$$H_z = C\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

$$H_x = -\frac{1}{h^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right)C\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right)$$

$$H_y = -\frac{1}{h^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right)C\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right)$$

$$E_x = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right)C\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

$$E_y = -\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right)C\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

$$h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2.$$

(a) (8 pts) Determine dominant modes and the lowest resonant frequencies for various cavity size conditions.

(b) (8 pts) For TE_{101}, please prove that the total time-average stored electric energy is equal to the total time-average stored maagnetic energy at the resonant frequency.

(c) (4 pts) Find the Quality factor of TE_{101}.

Part III

8. (35pts) An electron cloud of an atom is bound to the positive nucleus by an attractive electric force, following by Hook's law; $F = -kx, \omega_0 = \sqrt{k/m_e}$. As a time-varying electric field E(t) of the wave is incident on the atom, the atom can be regarded as a classical forced oscillator driven by the E(t) and suffering a damped force of a form $m_e\gamma dx/dt$.

(a) Please write down the force equation of a driven damped oscillator and explain the significance of each term.(9pts)

(b) Let $E = E_0 e^{iwt}$ and $x = x_0 e^{i(wt-\alpha)}$, where E_0 and x_0 are real quantities. Substitute into the above expression and show that $x_0 = \frac{q_e E_0}{m_e}\frac{1}{[(\omega_0{}^2-\omega^2)^2+\gamma^2\omega^2]^{\frac{1}{2}}}$ (5pts)

(c) Derive an expression for the phase lag α, and discuss how α varies as ω goes from $\omega<<\omega_0$ to $\omega=\omega_0$ to $\omega>>\omega_0$. (5pts)

(d) Based on (b), write down the polarization ***P*** and electric displacement ***D*** in terms of *x*, ε, and ε_0, if there are *N* contributing electrons per unit volume. (6pts)

(e) Based on (d), show that if the imaginary part κ of the complex index of refraction is much smaller than the real part *n*, then for the case of a single resonant frequency ω_0, the following approximation is valid: (10pts)

$$\mathrm{n} = 1 + \frac{Ne^2}{2m\epsilon_0}\left(\frac{1}{\omega_0^2 - \omega^2}\right) \qquad \kappa = \frac{Ne^2}{2m\epsilon_0}\left(\frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2}\right)$$

9. (15pts)The far-zone electric field of a linear dipole antenna with length 2h (as shown in Figure 2) is

$\vec{E} = \hat{a}_\theta \dfrac{jA}{R} F(\theta)\exp(-j2\pi R/\lambda)$ where A is constant, and F(θ) is defined as

$$F(\theta) = \frac{\cos(\frac{2\pi}{\lambda}h\cos\theta) - \cos(\frac{2\pi}{\lambda}h)}{\sin\theta}$$

(a) Find the far-zone magnetic field. (7pts)

(b) Calculate the total power radiated by a quarter-wave dipole (i.e., a dipole with h=l/8). In this part, if you are not able to evaluate the integral(s) you get, then just leave the integral(s) there. (8pts)

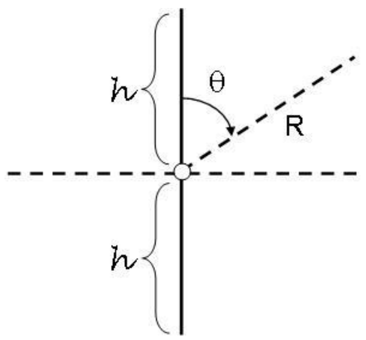

Figure 2